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WOLFF'S PROOF OF THE CORONA THEOREM[†]

BY

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ABSTRACT

An expository account is given of T. Wolff's recent elementary proof of Carleson's Corona Theorem (1962). The Corona Theorem answers affirmatively a question raised by S. Kakatani (1957) as to whether the open unit disc in the complex plane is dense in the maximal ideal space of the Banach algebra of bounded analytic functions thereon.

Recently T. Wolff discovered a simple proof of L. Carleson's Corona Theorem, asserting that the open unit disc $D = \{|z| < 1\}$ is dense in the maximal ideal space of the algebra $H^\infty(D)$. Here we give a version of Wolff's proof which does not depend on Carleson measures and BMO. There are three main ingredients to the proof:

(i) Elementary H^p-theory, including inner-outer factorization, and the identification of H_0^1 as the annihilator of the analytic polynomials in $C(\partial D)$.

(ii) Facility with the operators $\partial/\partial z$, $\partial/\partial\bar{z}$ and $\Delta = 4\partial^2/\partial z\partial\bar{z}$, together with the fact that the function

$$w(\zeta) = \frac{1}{\pi}\iint_{\{|z|\leqq 1+\varepsilon\}} \frac{u(z)}{z-\zeta}\,dxdy \tag{0.1}$$

satisfies the $\bar{\partial}$-equation

$$\frac{\partial w}{\partial \bar{z}} = u$$

on the domain of integration.

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(iii) Green's formula for the unit disc,

$$u(0)=\frac{1}{2\pi}\int_{\partial D} u\,d\theta-\frac{1}{2\pi}\iint_{D}(\Delta u)\log\frac{1}{|z|}\,dxdy, \tag{0.2}$$

valid for u smooth.

All H^p-spaces will be with respect to the unit disc D. The norm on H^p will be that of $L^p(d\theta/2\pi)$:

$$\|f\|_p=\left(\int |f(e^{i\theta})|^p\frac{d\theta}{2\pi}\right)^{1/p}.$$

The Corona Theorem will be established in the following equivalent form.

Corona Theorem [2, 3]. *For $n \geqq 1$ and $\delta > 0$, there exist constants $C(n,\delta)$ with the following property. If $f_1,\cdots,f_n \in H^\infty$ satisfy $|f_j|\leqq 1$, $1\leqq j\leqq n$, and $|f_1|^2+\cdots+|f_n|^2\geqq\delta$ on D, then there exist $g_1,\cdots,g_n\in H^\infty$ such that $f_1g_1+\cdots+f_ng_n=1$, and $|g_j|\leqq C(n,\delta)$, $1\leqq j\leqq n$.*

The proof will be broken into four parts.

Part I. Reduction to a $\bar\partial$-problem

By a normal family argument, we can assume that $f_1,\cdots,f_n$ are analytic across ∂D. Set

$$h_j=\bar f_j/(|f_1|^2+\cdots+|f_n|^2),\qquad 1\leqq j\leqq n.$$

Each h_j is C^∞, and $\sum f_jh_j=1$. However, since the h_j's are not analytic, they must be modified.

Let w_{jk} be smooth functions defined in a neighborhood of $\bar D$, and satisfying there the $\bar\partial$-equation

$$\frac{\partial w_{jk}}{\partial\bar z}=h_j\frac{\partial h_k}{\partial\bar z},\qquad 1\leqq j,k\leqq n. \tag{1.1}$$

Set

$$g_j=h_j+\sum_k(w_{jk}-w_{kj})f_k,\qquad 1\leqq j\leqq n.$$

Clearly $\sum f_jg_j=1$. Since $\sum f_jh_j=1$, also

$$\sum f_k\frac{\partial h_k}{\partial\bar z}=\frac{\partial}{\partial\bar z}\left(\sum f_kh_k\right)=0$$

and we find that

$$\frac{\partial g_j}{\partial \bar{z}} = \frac{\partial h_j}{\partial \bar{z}} + h_j \sum f_k \frac{\partial h_k}{\partial \bar{z}} - \frac{\partial h_j}{\partial \bar{z}} \sum f_k h_k = 0.$$

Thus the g_j's are analytic on $\bar{D}$. To complete the proof, it suffices now to find w_{jk}'s that satisfy (1.1) and that satisfy also an estimate of the form

$$\| w_{jk} \|_\infty \leqq C_0(n, \delta). \tag{1.2}$$

It may appear that the g_j's were pulled out of the proverbial hat. However, the technique for adjusting the h_j's is part of an established algebraic formalism, involving the Koszul complex. The idea of applying the formalism of the Koszul complex to the case at hand, and of reducing to estimates for the $\bar{\partial}$-operator, is due to L. Hörmander [7].

Part II. Solving a $\bar{\partial}$-problem with Bounds

Let u be a smooth function defined on a neighborhood of $\bar{D}$. We wish to find w such that $\partial w/\partial \bar{z} = u$, while $\| w \|_\infty$ is as small as possible.

By our preliminary remarks, there always exists a smooth function w_0 satisfying $\partial w_0/\partial \bar{z} = u$; we can take w_0 to be defined by (0.1). The general solution then has the form $w = w_0 + p$, where p is analytic. Hence

$$\inf\left\{ \| w \|_\infty : \frac{\partial w}{\partial \bar{z}} = u \text{ near } \bar{D} \right\} = \inf\{ \| w_0 + p \|_\infty : p \text{ analytic on } \bar{D}\}. \tag{2.1}$$

By duality, this latter infimum coincides with

$$\sup\left\{ \left| \int w_0 F \frac{d\theta}{2\pi} \right| : F \in H_0^1, \| F \|_1 \leqq 1 \right\}.$$

We may restrict our attention here to F analytic across ∂D. Since $F(0) = 0$, Green's formula (0.2) applied to $w_0 F$ yields

$$\int_{\partial D} w_0 F d\theta = \iint_D \Delta(w_0 F) \log \frac{1}{|z|} dxdy.$$

Now

$$\Delta(w_0 F) = 4 \frac{\partial^2}{\partial z \partial \bar{z}} (w_0 F) = 4 \frac{\partial}{\partial z} (uF) = 4F'u + 4F \frac{\partial u}{\partial z}.$$

Thus the quantity (2.1) can be reexpressed in the form

$$\frac{2}{\pi} \sup_{F \in H_0^1, \| F \|_1 \leqq 1} \left| \iint_D F' u \log \frac{1}{|z|} dxdy + \iint_D F \frac{\partial u}{\partial z} \log \frac{1}{|z|} dxdy \right|. \tag{2.2}$$

To complete the proof, it suffices now to bound the integrals appearing in (2.2) by constants $C_0(n,\delta)$, in the case that $u = h_j \partial h_k / \partial \bar{z}$. For this, we look for estimates for integrals of the form appearing in (2.2).

The idea of expressing (2.1) in the form (2.2) is Wolff's key contribution.

Part III. Estimates for Integrals

Suppose that g is harmonic in a neighborhood of $\bar{D}$. Applying Green's formula (0.2) to $u = |g|^2$, and noting that $\Delta u = 2|\nabla g|^2$, we obtain the fundamental identity of Littlewood–Paley theory:

$$\frac{1}{\pi}\iint_D |\nabla g|^2 \log\frac{1}{|z|}\,dxdy = \int_{\partial D} |g|^2 \frac{d\theta}{2\pi} - |g(0)|^2.$$

If furthermore g is analytic, then $|\nabla g|^2 = 2|g'|^2$, and the identity becomes

$$\frac{2}{\pi}\iint_D |g'|^2 \log\frac{1}{|z|}\,dxdy = \|g\|_2^2 - |g(0)|^2.$$

This yields immediately our basic estimate

$$\iint_D |g'|^2 \log\frac{1}{|z|}\,dxdy \leqq \frac{\pi}{2}\|g\|_2^2, \qquad g \in H^2, \tag{3.1}$$

which can also be obtained by developing g in a power series.

Lemma 1. *If $f_1, f_2 \in H^\infty$ and $g_1, g_2 \in H^2$, then*

$$\iint_D |g_1 g_2 f_1' f_2'| \log\frac{1}{|z|}\,dxdy \leqq 2\pi \|g_1\|_2 \|g_2\|_2 \|f_1\|_\infty \|f_2\|_\infty.$$

Proof. In view of the Cauchy–Schwarz inequality, it suffices to show that if $f \in H^\infty$ and $g \in H^2$, then

$$\iint_D |gf'|^2 \log\frac{1}{|z|}\,dxdy \leqq 2\pi \|g\|_2^2 \|f\|_\infty^2. \tag{3.2}$$

Since $gf' = (gf)' - g'f$, we have

$$|gf'|^2 \leqq 2[|(gf)'|^2 + |g'f|^2] \leqq 2|(gf)'|^2 + 2\|f\|_\infty^2 |g'|^2.$$

The integrals of both summands on the right are easily estimated by (3.1):

$$2\int |(gf)'|^2 \log\frac{1}{|z|}\,dxdy \leqq \pi \|gf\|_2^2 \leqq \pi \|g\|_2^2 \|f\|_\infty^2,$$

$$2\|f\|_\infty^2 \int |g'|^2 \log\frac{1}{|z|}\,dxdy \leqq \pi \|g\|_2^2 \|f\|_\infty^2.$$

Combining these estimates, we obtain (3.2). □

LEMMA 2. *If $F \in H^1$ and $f_1, f_2 \in H^\infty$, then*

$$\iint_D |Ff_1'f_2'| \log \frac{1}{|z|} dxdy \leqq 2\pi \|F\|_1 \|f_1\|_\infty \|f_2\|_\infty.$$

PROOF. Write $F = g_1 g_2$, where $g_1, g_2 \in H^2$ satisfy $\|g_1\|_2^2 = \|g_2\|_2^2 = \|F\|_1$, and apply Lemma 1. □

LEMMA 3. *If $f \in H^\infty$ and $g_1, g_2 \in H^2$, then*

$$\iint_D |g_1 g_2' f'| \log \frac{1}{|z|} dxdy \leqq \pi \|g_1\|_2 \|g_2\|_2 \|f\|_\infty.$$

PROOF. By the Cauchy–Schwarz inequality, the integral is estimated by

$$\left[\iint_D |g_2'|^2 \log \frac{1}{|z|} dxdy\right]^{1/2} \left[\iint_D |g_1 f'|^2 \log \frac{1}{|z|} dxdy\right]^{1/2}.$$

Now estimate the first factor using (3.1) and the second factor using Lemma 1, or (3.2). □

LEMMA 4. *If $F \in H^1$ and $f \in H^\infty$, then*

$$\iint_D |F'f'| \log \frac{1}{|z|} dxdy \leqq 2\pi \|F\|_1 \|f\|_\infty.$$

PROOF. As usual we can assume that both F and f are analytic on the closed disc $\bar{D}$, and we can assume furthermore that F has no zeros on ∂D. Then $F = g_1 g_2$, where g_1 and g_2 are analytic on $\bar{D}$, and $\|g_1\|_2^2 = \|g_2\|_2^2 = \|F\|_1$. Substituting $F' = g_1 g_2' + g_1' g_2$ and applying Lemma 3, we obtain the estimate of Lemma 4. □

Carleson measures and BMO enter the picture as follows. Consider the estimate (3.2). It asserts that the measure $|f'|^2 \log(1/|z|)dxdy$ determines a continuous linear functional on H^2. This is equivalent to the measure being a Carleson measure [6, chapter II]. For an analytic function f, say of class H^1, it turns out that $|f'|^2 \log(1/|z|)dxdy$ is a Carleson measure if and only if $f \in$ BMO [6, chapter VI], and there is a corresponding relation between the Carleson and BMO norms. Thus the supremum norm in (3.2) and elsewhere, can be replaced by the BMO-norm, providing the constants are adjusted accordingly.

Part IV. Conclusion of Proof

It suffices to estimate the integrals appearing in (2.2), in the case

$$u = h_j \frac{\partial h_k}{\partial \bar{z}},$$

and for this purpose we will use Lemmas 2 and 4. Writing $\varphi = 1/(\Sigma |f_m|^2)$, we compute

$$\frac{\partial h_k}{\partial \bar{z}} = \varphi \bar{f}'_k - \varphi^2 \bar{f}_k \sum f_m \bar{f}'_m,$$

$$u = \varphi^2 \bar{f}_j \bar{f}'_k - \varphi^3 \bar{f}_j \bar{f}_1 \sum f_m \bar{f}'_m,$$

$$\frac{\partial u}{\partial z} = -2\varphi^3 \bar{f}_j \bar{f}'_k \sum f'_m \bar{f}_m - \varphi^3 \bar{f}_j \bar{f}_k \sum |f'_m|^2 + 3\varphi^4 \bar{f}_j \bar{f}_k \left| \sum f_m f'_m \right|^2.$$

The expression for u depends linearly on the $\bar{f}'_m$'s. Thus we can use Lemma 4 to estimate the first integral appearing in (2.2). Taking into account $\|f_m\|_\infty \leqq 1$, $1 \leqq m \leqq n$, and $\varphi \leqq 1/\delta$, we bound the first integral in (2.2) by a constant of the form $c(n)/\delta^3$, where c depends only on n.

The expression for $\partial u/\partial z$ depends quadratically on the f'_m's and their complex conjugates. Thus we can use Lemma 2 to estimate the second integral in (2.2), and this time we obtain a bound of the form $c(n)/\delta^4$. This completes the proof of the Corona Theorem, and we see that we can take the constants $C(n, \delta)$ to be of the form $c(n)/\delta^4$.

More General Domains

Based on Carleson's Theorem, E. L. Stout [10] gave a proof of the Corona Theorem for finitely connected domains in the plane. M. Behrens [1] was first to discover a class of infinitely connected domains in the plane for which the Corona Theorem is valid. The problem for arbitrary domains in the plane remains unsolved.

It is proved in [4] that the version of the Corona Theorem stated above is valid for any finitely connected domain, with constants $C_m(n, \delta)$ that depend only on n, δ, and the connectivity m of the domain. The problem for arbitrary plane domains is equivalent to determining whether the best constants $C_m(n, \delta)$ remain bounded as $m \to \infty$. If the constants $C_m(n, \delta)$ are bounded, one obtains a solution of the equation $\Sigma f_j g_j = 1$ as a limit of solutions on finitely connected subdomains by a normal families argument. If the constants $C_m(n, \delta)$ are unbounded for some fixed $n \geqq 1$ and $\delta > 0$, then Behrens' construction yields domains of reasonably simple types for which the Corona Theorem fails. For instance, it would fail for a domain obtained from the punctured disc $D\backslash\{0\}$ by excising a sequence of disjoint closed subdiscs that converge to 0.

The Corona Theorem is valid for finite bordered Riemann surfaces, though B. Cole [5, chapter IV] has constructed a Riemann surface with a large collection of bounded analytic functions, for which the Corona Theorem fails. Cole's example can be modified to obtain a bounded pseudoconvex domain in $\mathbf{C}^3$, with smooth, strictly pseudoconvex boundary except at one point, for which the Corona Theorem fails. N. Sibony [9] has discovered a proper pseudoconvex subdomain V of the unit bidisc D^2 in $\mathbf{C}^2$ such that every bounded analytic function on V extends to be analytic on D^2. In particular, the Corona Theorem fails rather spectacularly for V. Meanwhile, it is not known whether the Corona Theorem is valid for a polydisc or ball in $\mathbf{C}^n$, $n \geqq 2$.

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